Domains and Ranges of Functions of Several Variables

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Overview

In studying real world phenomenon, a quantity being investigated usually depends on two or more independent variables. So we need to extend the basic ideas of the calculus of functions of a single variable to functions of several variables. Although the calculus rules remain essentially the same, the calculus is even richer.

The studies of probability, statistics, fluid dynamics, and electricity, to mention only a few, all lead in natural ways to functions of more than one variable.

We shall extend the ideas of single-variable differential calculus to functions of several variables. Their derivatives are more varied and interesting because of the different ways the variables can interact. The applications of these derivatives are also more varied than for single-variable calculus.

Overview

Many functions depend on more than one independent variable. The function $V = \pi r^2 h$ calculates the volume of a right circular cylinder from its radius and height. The function

$$f(x, y) = x^2 + y^2$$

calculates the height of the paraboloid $z = x^2 + y^2$ above the point $P(x, y)$ from the two coordinates of P.

The temperature T of a point on Earth's surface depends on its latitude x and longitude y, expressed by writing $T = f(x, y)$. In this lecture, we define functions of more than one independent variable and discuss ways to graph them.

Definition 1.

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real-valued function *f on D is a rule that assigns a unique (single) real number*

$$w = f(x_1, x_2, \ldots, x_n)$$

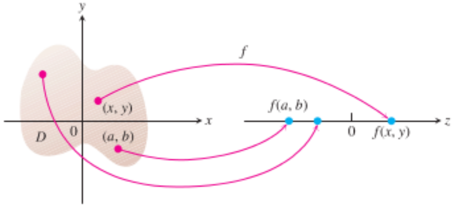
to each element in D. The set D is the function's domain*.*

The set of w-values taken on by f is the function's range*. The symbol w is the* dependent variable *of f, and f is said to be a function of the n* independent variable *x_1 to x_n. We also call the x_j's the function's* input variables *and call w the function's* output variable*.*

Functions of Two Independent Variables

If f is a function of two independent variables, we usually call the independent variables x and y and picture the domain of f as a region in the xy-plane. If f is a function of three independent variables, we call the variables x, y, and z and picture the domain as a region in space.

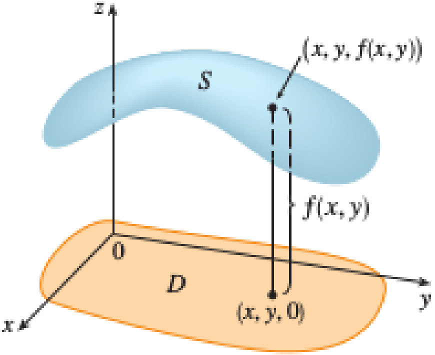
As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable.



An arrow diagram for the function $z = f(x, y)$

Functions of Two Independent Variables

If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^2$ such that $z = f(x, y)$ and (x, y) is in D.



Domains and Ranges

In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If

$$f(x,y) = \sqrt{y - x^2},$$

then y cannot be less than x^2.

If

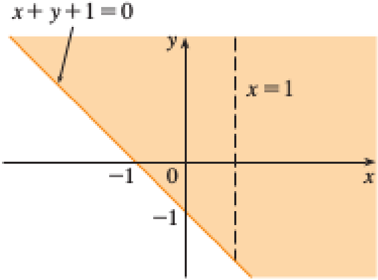
$$f(x,y) = \frac{1}{xy}$$

then xy cannot be zero. **The domain of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly.** The range consists of the set of output values for the dependent variable.

Example : Functions of Two Variables

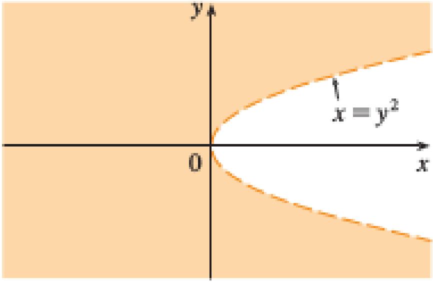
Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0,\infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	[-1,1]

Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$



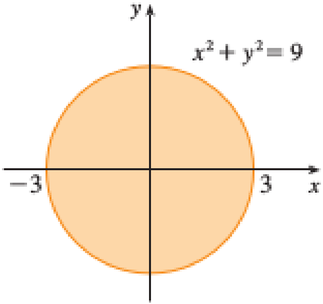
Domain of $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$

Domain of $f(x, y) = x \ln(y^2 - x)$



Domain of $f(x, y) = x \ln(y^2 - x)$

Domain of $f(x, y) = \sqrt{9 - x^2 - y^2}$



Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Example : Functions of Three Variables

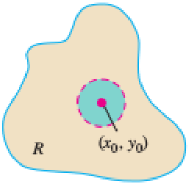
Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0,\infty)$
$w = \frac{1}{x^2+y^2+z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty,\infty)$

Interior Point and Interior of a Region

Regions in the plane can have interior points just like intervals on the real line.

Definition 2.

A point (x_0, y_0) in a region (set) R in the xy-plane is an interior point *of R if it is the center of a disk of positive radius that lies entirely in R. The interior points of a region, as a set, make up the* interior *of the region. An interior point is necessarily a point of R.*



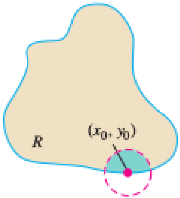
Interior point

Boundary Point and Boundary of a Region

Regions in the plane can have boundary points just like intervals on the real line.

Definition 3.

A point (x_0, y_0) *is a* boundary point *of* R *if every disk centered at* (x_0, y_0) *contains points that lie outside of* R *as well as points that lie in* R*. (The boundary point itself need not belong to* R*.) The region's boundary points make up its* boundary*. A boundary point of* R *need not belong to* R*.*



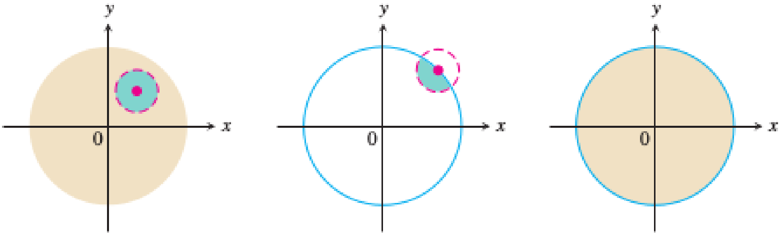
Boundary point

Definition 4.

A region is open *if it consists entirely of interior points. A region is* closed *if it contains all its boundary points.*

As with a half-open interval of real numbers $[a, b)$, some regions in the plane are neither open nor closed. If you start with the open disk and add to it some of but not all its boundary points, the resulting set is neither open nor closed. The boundary points that are there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

Interior Points and Boundary Points of the Unit Disk



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all boundary points.

Interior points and boundary points of the unit disk in the plane.

Definition 5.

A region in the plane is bounded *if it lies inside a disk of fixed radius. A region is* unbounded *if it is not bounded.*

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks.

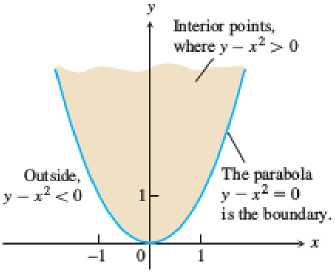
Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

Example

The domain of

$$f(x, y) = \sqrt{y - x^2}$$

consists of the shaded region and its bounding parabola $y = x^2$.



The domain of
$f(x, y) = \sqrt{y - x^2}$ consists of the shaded
region and its bounding parabola $y = x^2$

Graphs, Level Curves, and Contours of Functions of Two Variables

There are **two standard ways** to picture the values of a function $f(x, y)$. One is to draw and label curves in the domain on which f has a constant value. The other is to sketch the surface $z = f(x, y)$ in space.

Definition 6 (Level Curve, Graph, Surface).

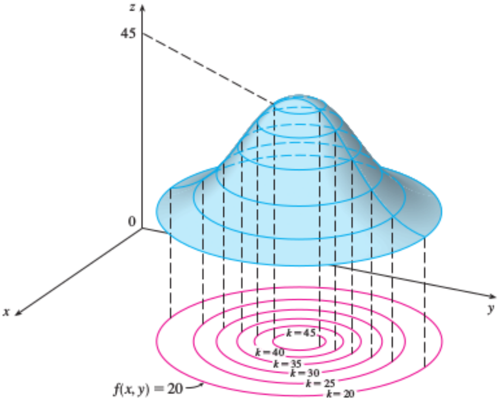
The set of points in the plane where a function $f(x, y)$ has a constant value

$$f(x, y) = c$$

is called a level curve *of f.*

The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the graph *of f. The graph of f is also called the* surface *$z = f(x, y)$.*

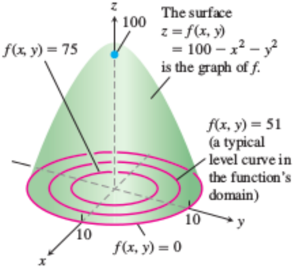
Showing level curves being lifted up to graphs of functions.



Example

The graph and selected level curves of the function

$$f(x, y) = 100 - x^2 - y^2.$$



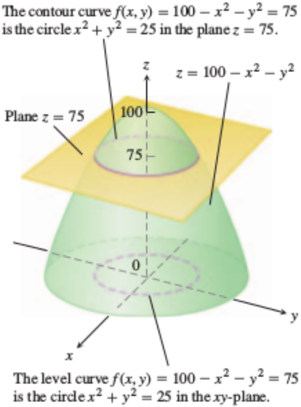
The graph and selected
level curves of the function
$f(x, y) = 100 - x^2 - y^2$.

Contour Curves

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$.

It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f.

Level and Contour Curves



A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

The figure shows the contour curve $f(x, y) = 75$ on the surface

$$z = 100 - x^2 - y^2$$

defined by the function

$$f(x, y) = 100 - x^2 - y^2.$$

The contour curve lies directly above the circle

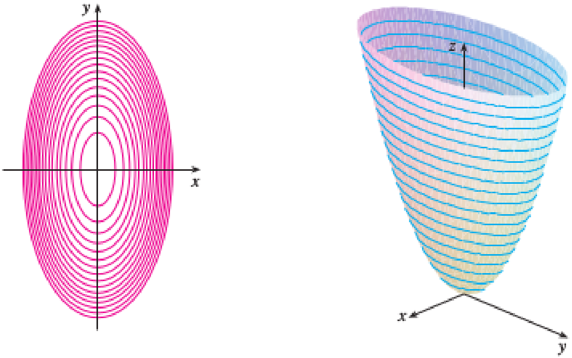
$$x^2 + y^2 = 25,$$

which is the level curve

$$f(x, y) = 75$$

in the function's domain.

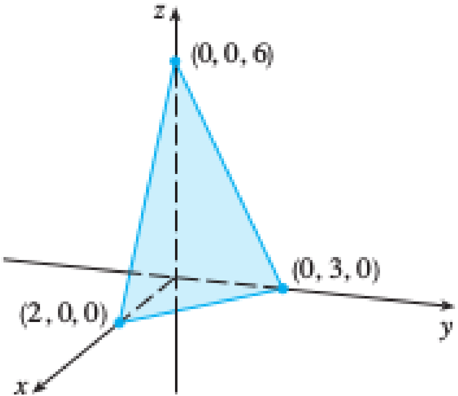
The graph of $h(x, y) = 4x^2 + y^2$ is formed by lifting the level curves.



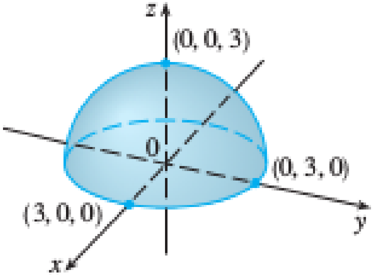
(a) Contour map

(b) Horizontal traces are raised level curves

The graph of the function $f(x, y) = 6 - 3x - 2y$.

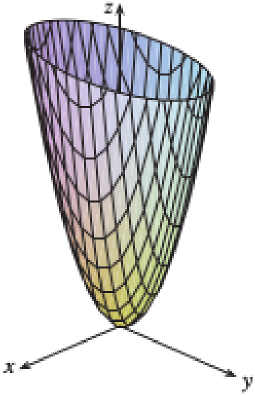


The graph of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$.

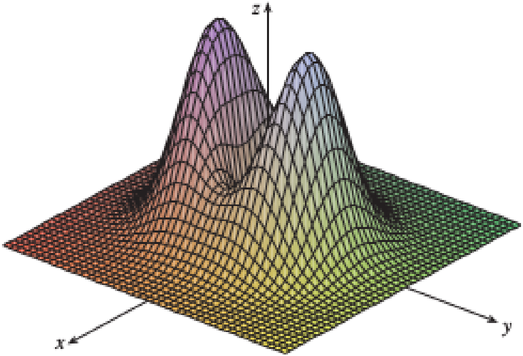


Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

The graph of the $h(x, y) = 4x^2 + y^2$.

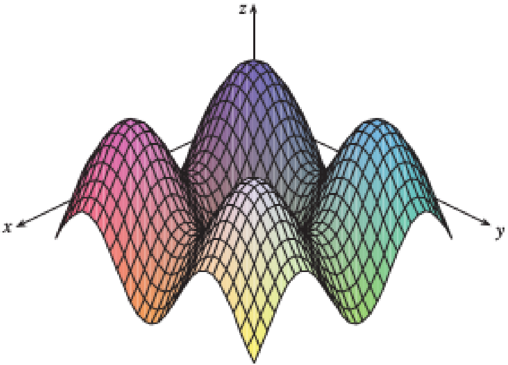


The graph of the $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$.



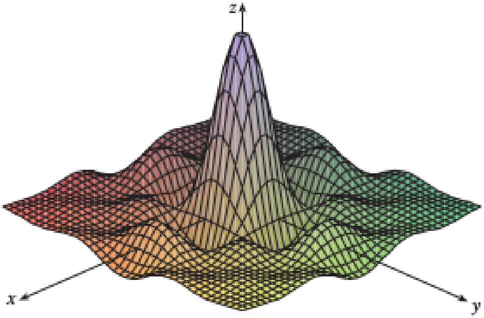
$f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

The graph of the $f(x, y) = \sin x + \sin y$.



$f(x, y) = \sin x + \sin y$

The graph of the $f(x, y) = \dfrac{\sin x \sin y}{xy}$.



$$f(x, y) = \frac{\sin x \sin y}{xy}$$

Level Surfaces

In the plane, the points where a function of two independent variables has a constant value $f(x, y) = c$ make a curve in the function's domain.

In space, the points where a function of three independent variables has a constant value $f(x, y, z) = c$ make a surface in the function's domain.

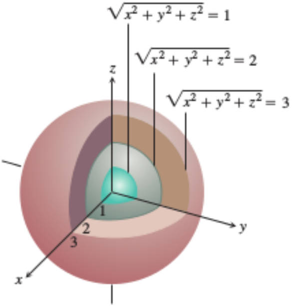
Definition 7.

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a level surface *of f.*

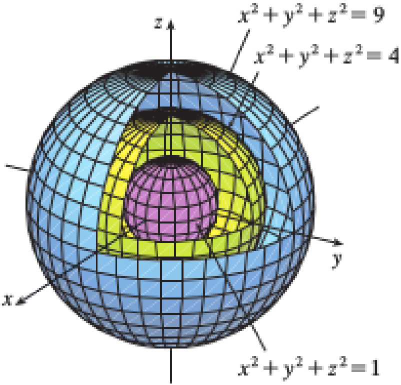
Level Surfaces

Since the graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ **lying in a four-dimensional space**, we cannot sketch them effectively in our three-dimensional frame of reference. We can see how the function behaves, however, by looking at its three-dimensional level surfaces.

The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres.



Level Surfaces

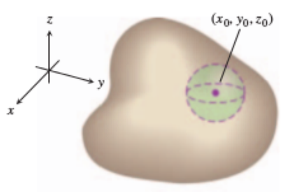


Interior Points for Space Regions

The definitions of interior, boundary, open, closed, bounded, and unbounded for regions in space are similar to those for regions in the plane. To accommodate the extra dimension, we use solid balls of positive radius instead of disks.

Definition 8.

A point (x_0, y_0, z_0) *in a region* R *in space is an* interior point *of* R *if it is the center of a solid ball that lies entirely in* R. *The* interior *of* R *is the set of interior points of* R.

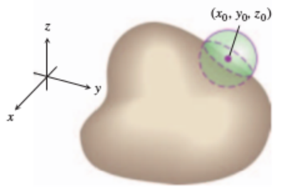


A region is open *if it consists entirely of interior points.*

Boundary Points for Space Regions

Definition 9.

A point (x_0, y_0, z_0) is a boundary point *of R if every sphere centered at (x_0, y_0, z_0) encloses points that lie outside of R as well as points that lie inside R. The* boundary *of R is the set of boundary points of R.*



A region is closed *if it contains its entire boundary.*

Examples

Examples of *open* sets in space include the interior of a sphere, the open half-space $z > 0$, the first octant (where x, y, and z are all positive), and space itself.

Examples of *closed* sets in space include lines, planes, the closed half-space $z \geq 0$, the first octant together with its bounding planes, and space itself (since it has no boundary points).

A solid sphere with part of its boundary removed or a solid cube with a missing face, edge, or corner point would be *neither open nor closed*.

Functions of more than three independent variables

Functions of more than three independent variables are also important.

For example, the temperature on a surface in space may depend not only on the location of the point $P(x, y, z)$ on the surface, but also on time t when it is visited, so we would write $T = f(x, y, z, t)$.

Modeling Temperature Beneath Earth's Surface

Example 10.

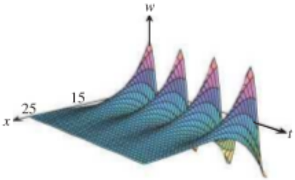
The temperature beneath the Earth's surface is a function of the depth x beneath the surface and the time t of the year. If we measure x in feet and t as the number of days elapsed from the expected date of the yearly highest surface temperature, we can model the variation in temperature with the function

$$w = \cos(1.7 \times 10^{-2} t - 0.2x) e^{-0.2x}$$

(The temperature at 0 ft is scaled to vary from $+1$ to -1, so that the variation at x feet can be interpreted as a fraction of the variation at the surface.)

Modeling Temperature Beneath Earth's Surface

The following figure shows a computer-generated graph of the function. At a depth of 15 ft, the variation (change in vertical amplitude in the figure) is about 5% of the surface variation. At 30 ft, there is almost no variation during the year.



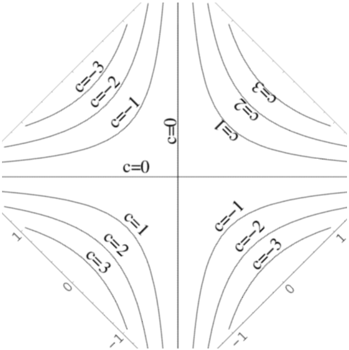
Modeling Temperature Beneath Earth's Surface

It also shows the seasonal variation of the temperature belowground as a fraction of surface temperature. At $x = 15$ ft, the variation is only 5% of the variation at the surface. At $x = 30$ ft, the variation is less than 0.25% of the surface variation.

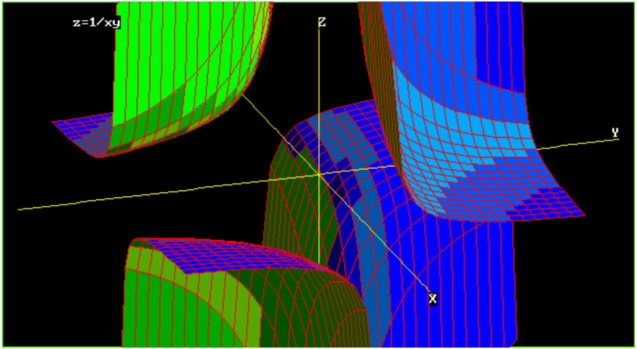
The graph also shows that the temperature 15 ft below the surface is about half a year out of phase with the surface temperature.

When the temperature is lowest on the surface (late January, say), it is at its highest 15 ft below. Fifteen feet below the ground, the seasons are reversed.

Level Curves for $f(x, y) = 1/xy$



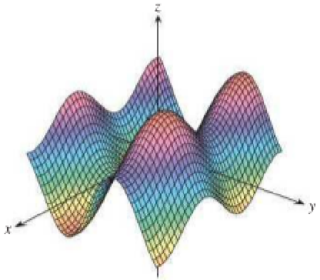
Graph of $f(x, y) = 1/xy$

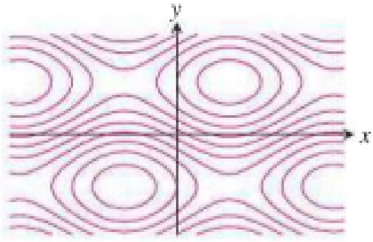


Computer-generated Graphs and Level Curves of Typical Functions of Two Variables

Computer-generated Graphs and Level Curves of

$$z = \sin x + 2 \sin y$$

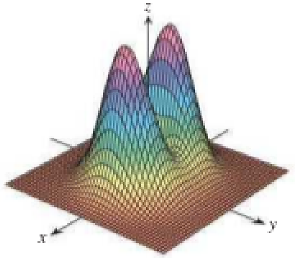


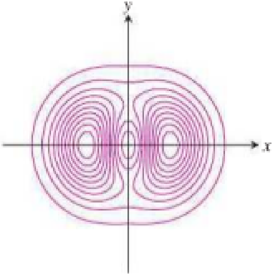


Computer-generated Graphs and Level Curves of Typical Functions of Two Variables

Computer-generated Graphs and Level Curves of

$$z = (4x^2 + y^2)e^{-x^2-y^2}$$

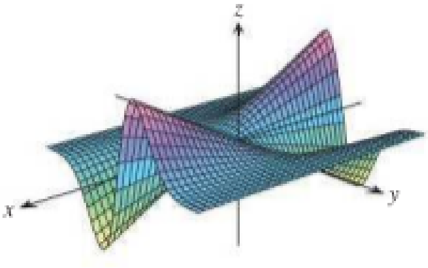


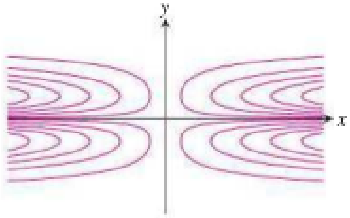


Computer-generated Graphs and Level Curves of Typical Functions of Two Variables

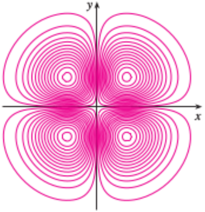
Computer-generated Graphs and Level Curves of

$$z = xye^{-y^2}$$

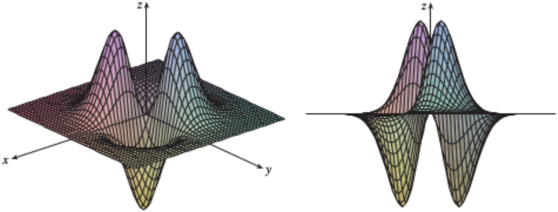




Computer-generated Graphs and Level Curves of Typical Functions of Two Variables

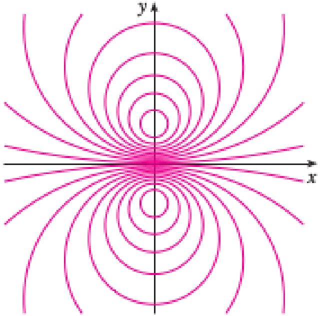


(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

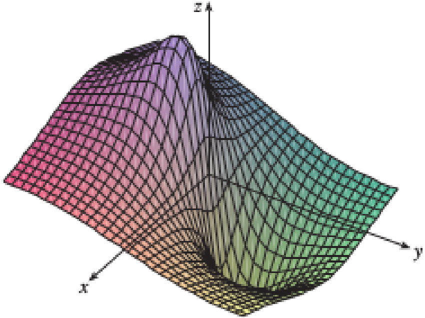


(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

Computer-generated Graphs and Level Curves of Typical Functions of Two Variables



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

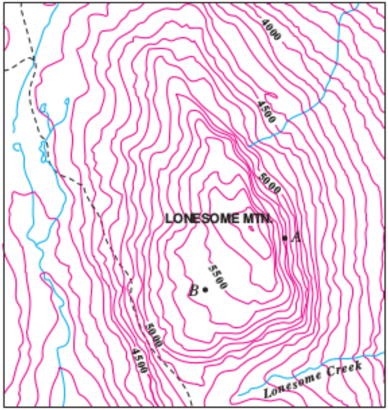
Topographic Maps of Mountainous Regions

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in the Figure. The level curves are curves of constant elevation above sea level. **If you walk along one of these contour lines, you neither ascend nor descend.**

Another common example is the temperature function — the temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

Weather map of the world indicating the average January temperatures

Here the level curves are called **isothermals** and join locations with the same temperature. Figure shows a weather map of the world indicating the average January temperatures. The isothermals are the curves that separate the colored bands.



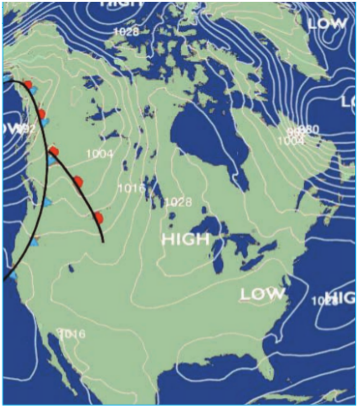
Weather map of the world indicating the average January temperatures

Functions of two variables can be visualized by means of level curves, which connect points where the function takes on a given value. Atmospheric pressure at a given time is a function of longitude and latitude and is measured in millibars.

Here the level curves are called **isobars** and those pictured join locations that had the same pressure on March 7, 2007. (The curves labeled 1028, for instance, connect points with pressure 1028 mb.) Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure, and are strongest where the isobars are tightly packed.

Atmospheric pressure map

The isobars in the atmospheric pressure map in the figure provide another example of level curves.



Exercise 11.

In the following exercises,

(a) *find the function's domain,*
(b) *find the function's range,*
(c) *describe the function's level curves,*
(d) *find the boundary of the function's domain,*
(e) *determine if the domain is an open region, a closed region, or neither, and*
(f) *decide if the domain is bounded or unbounded.*

1. $f(x, y) = \sqrt{y - x}$
2. $f(x, y) = \frac{1}{\sqrt{16 - x^2 - y^2}}$
3. $f(x, y) = \ln(x^2 + y^2)$
4. $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

Solution for (1.) in Exercise 11

(a) Domain: set of all (x, y) so that $y - x \geq 0 \Rightarrow y \geq x$

(b) Range: $z \geq 0$

(c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$

(d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line

(e) closed

(f) unbounded

Solution for (2.) in Exercise 11

(a) Domain: all (x, y) satisfying $x^2 + y^2 < 16$

(b) Range: $z \geq \frac{1}{4}$

(c) level curves are circles centered at the origin with radii $r < 4$

(d) boundary is the circle $x^2 + y^2 = 16$

(e) open

(f) bounded

Solution for (3.) in Exercise 11

(a) Domain: $(x, y) \neq (0, 0)$

(b) Range: all real numbers

(c) level curves are circles with center $(0, 0)$ and radii $r > 0$

(d) boundary is the single point $(0, 0)$

(e) open

(f) unbounded

Solution for (4.) in Exercise 11

(a) Domain: all (x, y), $x \neq 0$

(b) Range: $-\frac{\pi}{2} < z < \frac{\pi}{2}$

(c) level curves are the straight lines of the form $y = cx$, c any real number and $x \neq 0$

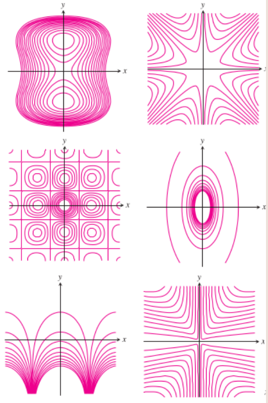
(d) boundary is the line $x = 0$

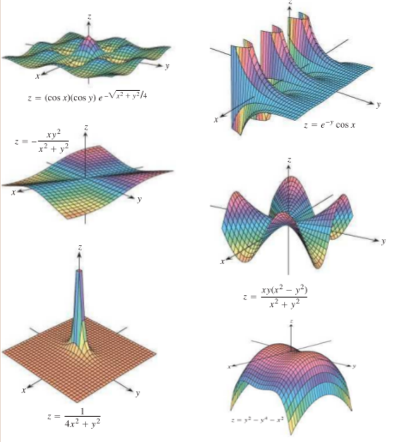
(e) open

(f) unbounded

Exercise 12.

Match each set of level curves with the appropriate function graphed.

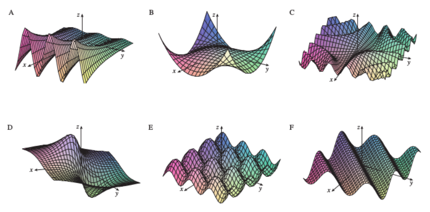


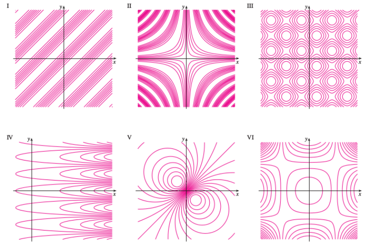


Identifying Surfaces and Level Curves

Exercise 13.

Match each set of level curves with the appropriate function graphed.





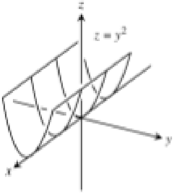
Exercise 14.

Display the values of the functions in the following exercises in two ways:
(a) by sketching the surface $z = f(x, y)$ *and*
(b) by drawing an assortment of level curves in the function's domain.
Label each level curve with its function value.

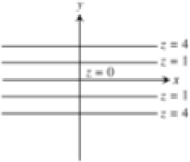
1. $f(x, y) = y^2$
2. $f(x, y) = \sqrt{x^2 + y^2}$
3. $f(x, y) = 1 - |y|$
4. $f(x, y) = 1 - |x| - |y|$

Solution for (1.) in Exercise 14

(a)

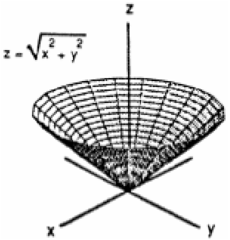


(b)

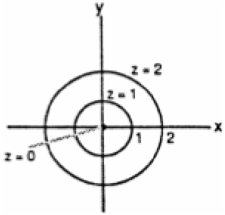


Solution for (2.) in Exercise 14

(a)

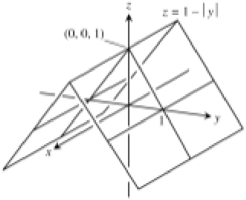


(b)

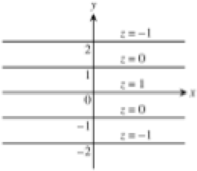


Solution for (3.) in Exercise 14

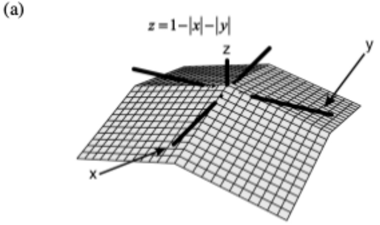
(a)

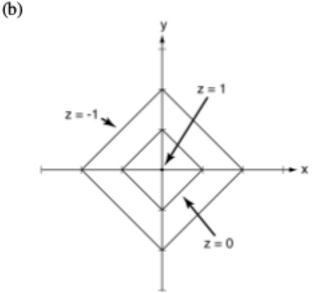


(b)



Solution for (4.) in Exercise 14





Finding a Level Curve

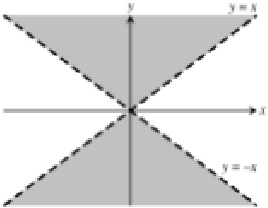
Exercise 15.

In the following exercises, find an equation for the level curve of the function $f(x, y)$ that passes through the given point.

1. $f(x, y) = 16 - x^2 - y^2$, $(2\sqrt{2}, \sqrt{2})$
2. $f(x, y) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$, $(1, 2)$

Solution for Exercise 15

1. Given that $f(x, y) = 16 - x^2 - y^2$.
   At $(2\sqrt{2}, \sqrt{2})$, $z = 16 - (2\sqrt{2})^2 - (\sqrt{2})^2 = 6$.
   So, $6 = 16 - x^2 - y^2$, hence $x^2 + y^2 = 10$.
2. $f(x, y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n = \frac{1}{1-\left(\frac{x}{y}\right)} = \frac{y}{y-x}$ for $\left|\frac{x}{y}\right| < 1$.
   Domain: all points (x, y) satisfying $|x| < |y|$.
   At $(1, 2)$, since $\left|\frac{1}{2}\right| < 1$, so $z = \frac{2}{2-1} = 2$.
   Hence $\frac{y}{y-x} = 2$, thus $y = 2x$.



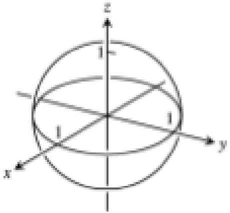
Sketching Level Surfaces

Exercise 16.

In the following exercises, sketch a typical level surface for the function.

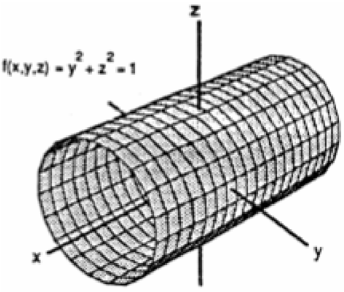
1. $f(x, y, z) = x^2 + y^2 + z^2$
2. $f(x, y, z) = y^2 + z^2$
3. $f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9)$

Solution for (1.) in Exercise 16

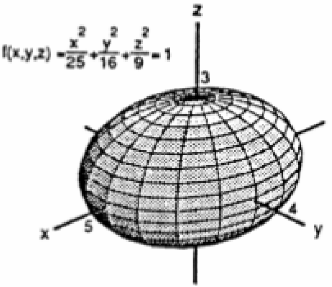


$f(x, y, z) = x^2 + y^2 + z^2 = 1$

Solution for (2.) in Exercise 16



Solution for (3.) in Exercise 16



Exercise 17.

In the following exercises, find an equation for the level surface of the function through the given point.

1. $f(x, y, z) = \ln(x^2 + y + z^2),\ (-1, 2, 1)$
2. $g(x, y, z) = \displaystyle\int_x^y \frac{d\theta}{1+\theta^2} + \int_0^z \frac{d\theta}{\sqrt{4-\theta^2}},\ (0, 1, \sqrt{3})$

Solution for Exercise 17

1. $f(x,y,z) = \ln(x^2 + y + z^2)$.
   At $(-1,2,1)$, $w = \ln(1+2+1) = \ln 4$, hence $\ln 4 = \ln(x^2 + y + z^2)$, thus $x^2 + y + z^2 = 4$.
2. $g(x,y,z) = \displaystyle\int_x^y \frac{dt}{1+t^2} + \int_0^x \frac{d\theta}{\sqrt{4-\theta^2}} = \tan^{-1} y - \tan^{-1} x + \sin^{-1}(\frac{z}{2})$.
   Domain: all points (x,y,z) satisfying $-2 \leq z \leq 2$.
   At $(0,1,\sqrt{3})$, $w = \tan^{-1} 1 - \tan^{-1} 0 + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{7\pi}{12}$.
   Hence $\tan^{-1} y - \tan^{-1} x + \sin^{-1}(\frac{z}{2}) = \frac{7\pi}{12}$.
   Since $-\frac{\pi}{2} \leq \sin^{-1}(\frac{z}{2}) \leq \frac{\pi}{2}$, $\frac{\pi}{12} \leq \tan^{-1} y - \tan^{-1} x \leq \frac{13\pi}{12}$, so
   $z = 2\sin(\frac{7\pi}{12} - \tan^{-1} y + \tan^{-1} x)$, where
   $\frac{\pi}{12} \leq \tan^{-1} y - \tan^{-1} x \leq \frac{13\pi}{12}$.

Exercise 18.

1. The maximum value of a function on a line in space :
   Does the function $f(x, y, z) = xyz$ *have a maximum value on the line* $x = 20 - t, y = t, z = 20$*? If so, what is it? Give reasons for your answer.*

   Hint : Along the line, $w = f(x, y, z)$ *is a differentiable function of* t.
2. The minimum value of a function on a line in space :
   Does the function $f(x, y, z) = xy - z$ *have a minimum value on the line* $x = t - 1, y = t - 2, z = t + 7$*? If so, what is it? Give reasons for your answer.*

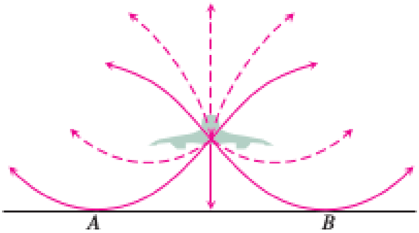
   Hint : Along the line, $w = f(x, y, z)$ *is a differentiable function of* t.

Solution for Exercise 18

1. $f(x, y, z) = xyz$ and $x = 20 - t$, $y = t, z = 20$.
   $w = (20 - t)(t)(20)$ along the line.
   $w = 400t - 20t^2 \Rightarrow \frac{dw}{dt} = 400 - 40t$;
   $\frac{dw}{dt} = 0 \Rightarrow 400 - 40t = 0 \Rightarrow t = 10$ and $\frac{d^2w}{dt^2} = -40$ for all t.
   Hence, maximum is at $t = 10$, so $x = 20 - 10 = 10$, $y = 10$, $z = 20$.
   Thus maximum of f along the line is
   $f(10, 10, 20) = (10)(10)(20) = 2000$.
2. $f(x, y, z) = xy - z$ and $x = t - 1$, $y = t - 2$, $z = t + 7$.
   $w = (t - 1)(t - 2) - (t + 7) = t^2 - 4t - 5$ along the line.
   $\frac{dw}{dt} = 2t - 4$; $\frac{dw}{dt} = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$ and $\frac{d^2w}{dt^2} = 2$ for all t.
   Hence maximum attains at $t = 2$, so $x = 2 - 1 = 1$, $y = 2 - 2 = 0$,
   and $z = 2 + 7 = 9$.
   Thus maximum of f along the line is $f(1, 0, 9) = (1)(0) - 9 = -9$.

The Concorde's sonic booms : Exercise

Sound waves from the *Concorde* bend as the temperature changes above and below the altitude at which the plane flies. The sonic boom carpet is the region on the ground that receives shock waves directly from the plane, not reflected from the atmosphere or diffracted along the ground. The carpet is determined by the grazing rays striking the ground from the point directly under the plane. (See accompanying figure.)



Sonic boom carpet

The Concorde's sonic booms : Exercise (contd...)

The width w of the region in which people on the ground hear the *Concorde's* sonic boom directly, not reflected from a layer in the atmosphere, is a function of

$T =$ air temperature at ground level (in degrees Kelvin)

$h =$ the Concorde's altitude (in kilometers)

$d =$ the vertical temperature gradient (temperature drop in degrees Kelvin per kilometer).

The formula for w is

$$w = 4\left(\frac{Th}{d}\right)^{1/2}.$$

The Concorde's sonic booms : Exercise (contd...)

The Washington-bound Concorde approached the United States from Europe on a course that took it south of Nantucket Island at an altitude of 16.8 km.

If the surface temperature is 290K and the vertical temperature gradient is 5 K/km, how many kilometers south of Nantucket did the plane have to be flown to keep its sonic boom carpet away from the island?

(From "Concorde Sonic Booms as an Atmospheric Probe" by N. K. Balachandra, W. L. Donn, and D. H. Rind, Science, Vol. 197 (July 1, 1977), pp 47-49.)

Exercise 19.

As you know, the graph of a real-valued function of a single real variable is a set in a two-coordinate space. The graph of a real-valued function of two independent real variables is a set in a three-coordinate space. The graph of a real-valued function of three independent real variables is a set in a four-coordinate space. How would you define the graph of a real-valued function $f(x_1, x_2, x_3, x_4)$ *of four independent real variables? How would you define the graph of a real-valued function* $f(x_1, x_2, x_3, \ldots, x_n)$ *of* n *independent real variables?*

Old Questions

Question 20.

Equation of the level surface of

$$f(x, y, z) = \sqrt{x - y} - \log z$$

passing through the point $(2, 1, e^2)$ *is*

(A) $f(x, y, z) = -1$

(B) $f(x, y, z) = 2$

(C) $f(x, y, z) = -2$

(D) $f(x, y, z) = 1$

(E) *None of the other options.*

The correct answer is
$f(x, y, z) = -1$.

Question 21.

Let $f(x,y) = \frac{\log(y-x)}{\sqrt{1+x-y}}$. *If* D *denotes the domain of* $f(x,y)$ *and* ∂D *denotes the Boundary of* D *then*

(A) D *is open and unbounded.*

(B) $D \bigcap \partial D = \emptyset$

(C) D *is neither open nor closed.*

(D) $D \bigcap \partial D \neq \emptyset$.

(E) *None of the other options*

The correct answers are
D *is open and unbounded* ; $D \bigcap \partial D = \emptyset$.

Question 22.

Let $f(x,y) = \log(y-x)\sqrt{\frac{1}{2}+x-y}$. *If* D *denotes the domain of* $f(x,y)$ *and* ∂D *denotes the Boundary of* D *then*

(A) D *is open and unbounded.*

(B) $D \bigcap \partial D = \emptyset$.

(C) D *is neither open nor closed.*

(D) $D \bigcap \partial D \neq \emptyset$.

(E) *None of the other options.*

The correct answers are
D is neither open nor closed ; $D \bigcap \partial D \neq \emptyset$.

Question 23.

Let $f(x, y) = \frac{\sqrt{1+x-y}}{\log(y-x)}$*. If* D *denotes the domain of* $f(x, y)$ *and* ∂D *denotes the Boundary of* D *then*

(A) D *is open and unbounded.*

(B) $D \bigcap \partial D = \emptyset$*.*

(C) D *is neither open nor closed.*

(D) $D \bigcap \partial D \neq \emptyset$*.*

(E) *None of the other options.*

The correct answers are
D is open and unbounded ; $D \bigcap \partial D = \emptyset$.

Question 24.

Let $f(x,y) = \frac{\sqrt{\frac{1}{2}+x-y}}{\log(y-x)}$. *If* D *denotes the domain of* $f(x,y)$ *and* ∂D *denotes the Boundary of* D *then*

(A) D *is open and unbounded.*

(B) $D \bigcap \partial D = \emptyset$.

(C) D *is neither open nor closed.*

(D) $D \bigcap \partial D \neq \emptyset$.

(E) *None of the other options.*

The correct answers are
D is neither open nor closed ; $D \bigcap \partial D \neq \emptyset$.

Question 25.

Let $f(x,y) = \sqrt{1-x}\log(y^2 - x)$*. If* D *denotes the domain of* $f(x,y)$ *and* ∂D *denotes the boundary of* D *then*

(A) D *is open and unbounded.*

(B) $\partial D \neq \emptyset$.

(C) D *is neither open nor closed.*

(D) $D \bigcap \partial D = \emptyset$.

(E) *None of the other options.*

The correct answers are
$\partial D \neq \emptyset$; D is neither open nor closed.

Question 26.

Which of the following functions has/have closed domain?

(A) $\frac{1}{xy}$

(B) $x^2 + y^2$

(C) $\log(y - x)\sqrt{\frac{1}{2} + x - y}$

(D) $\sqrt{x^2 + y^2 - z}$

(E) *None of the other options.*

The correct answers are
$x^2 + y^2$; $\sqrt{x^2 + y^2 - z}$.

Question 27.

Which of the following functions has/have open domain?

(A) $\frac{1}{xy}$

(B) $x^2 + y^2$

(C) $\sqrt{x-1}\log(y^2 - x)$

(D) $\sqrt{x^2 + y^2 - 4z}$

(E) *None of the other options.*

The correct answers are
$\frac{1}{xy}$; $x^2 + y^2$.

Old Questions

Question 28.

For the given function $f(x, y) = \sqrt{16 - x^2 - y^2}$*, the point* $(0, 0)$ *is on the level curve obtained by setting*

(A) $f(x, y) = 0$.

(B) $f(x, y) = 2$.

(C) $f(x, y) = 3$.

(D) $f(x, y) = 4$.

(E) *None of the other options.*

The correct answer is
$f(x, y) = 4$.

Old Questions

Question 29.

Let $f(x, y) = \sqrt{3x^2 + 2y^2}$. *Then the equation of the level curve passing through the point* $(2, -3)$ *is*

(A) $\frac{x^2}{10} + \frac{y^2}{15} = 1$

(B) $\frac{x^2}{6} + \frac{y^2}{9} = 1$

(C) $\frac{x^2}{4} + \frac{y^2}{6} = 1$

(D) $\frac{x^2}{12} + \frac{y^2}{18} = 1$

(E) *None of the other options.*

The correct answer is
$\frac{x^2}{10} + \frac{y^2}{15} = 1$.

References

1. M.D. Weir, J. Hass and F.R. Giordano, Thomas' Calculus, 11th Edition, Pearson Publishers.
2. N. Piskunov, Differential and Integral Calculus, Vol I & II (Translated by George Yankovsky).
3. S.C. Malik and Savitha Arora, Mathematical Analysis, New Age Publishers.
4. R. G. Bartle, D. R. Sherbert, Introduction to Real Analysis, Wiley Publishers.